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Symmetry algebras for superintegrable systems*

C Gonera, P Kosiński, M Majewski and P Maślanka

Department of Theoretical Physics II, University of Łódź, Pomorska 149/153, 90-236 Łódź, Poland

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Abstract

It is shown that the symmetry algebra of quantum maximally superintegrable systems can always be chosen to be $u(N)$, N being the number of degrees of freedom.

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A classical dynamical system of N degrees of freedom is called maximally superintegrable if it admits $2N - 1$ independent, globally defined integrals of motion. They form the complete set of integrals and any other integral of motion can be expressed in terms of them. In particular, the Poisson bracket of two basic integrals, being again a constant of motion, is their function (in general, nonlinear). We conclude that the integrals of motion for maximally superintegrable systems form a finite W-algebra [1, 2].

An interesting question arises; whether this algebra can be linearized to Lie algebra by a judicious choice of basic integrals. The answer to this question, for a large class of confining systems, admitting action-angle variables is positive [3] (see [4] for older papers on the subject, both in classical and quantum theory). Moreover, assuming that *all* action variables do appear in the Hamiltonian, the Lie algebra obtained is universally $u(N)$. The price one has to pay for this result is that the resulting integrals are usually quite complicated functions of basic dynamical variables.

The aim of the present paper is to extend this result to the quantum case. The first problem which arises when passing to the quantum case is the very definition of quantum maximally superintegrable systems. We shall use the following one which seems to cover most interesting cases. Assume we have a set of commuting selfadjoint operators $\hat{I}_k$, $k = 1, \ldots, N$ (quantum actions), such that:

(i) the spectrum of each $\hat{I}_k$ consists of the eigenvalues of the form $n_k + \sigma_k$, where $n_k \in N$ is any natural number and $\sigma_k \in \mathbb{R}$ is fixed;

(ii) the common eigenvectors of all $\hat{I}_k$, $|n\rangle$, $n \equiv (n_1, \ldots, n_N)$, span the whole space of states;

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(iii) the Hamiltonian can be written as

$$\hat{H} = H\left(\sum_{k=1}^{N} m_k \hat{I}_k\right) \tag{1}$$

with $m_k \in N, k = 1, \ldots, N$; we shall assume that all $m_k \neq 0$. Let us note that we can also assume that $m_1, \ldots, m_N$ have no common divisor except unity.

We believe that the above definition is flexible enough to cover known cases. Sometimes it may appear necessary to take an orthogonal sum of the few Hilbert spaces described above to cover the whole space of states. This can be easily understood by considering systems having classical counterparts. Then the classical limits of our $\hat{I}_k$ operators are expected to give action variables. On the other hand, classical actions are assumed to be nonnegative (which, in turn, agrees with our assumption concerning the spectra of $\hat{I}$'s). It happens sometimes that, due to the positivity condition, the action variables are given by the absolute values of relevant integrals of motion. The simplest example seems to be the planar Kepler problem where $I_\phi = |L|$, L being the angular momentum. On the quantum level this can be dealt with by considering separately the Hilbert subspaces corresponding to positive versus negative eigenvalues of angular momentum.

The vectors $|n\rangle$ are eigenvectors of $\hat{H}$:

$$\hat{H}|n\rangle = E_n|n\rangle \qquad E_n = H\left(\sum_{k=1}^{N} m_k n_k + \sum_{k=1}^{N} m_k \sigma_k\right). \tag{2}$$

We see that the energy spectrum is degenerate: all eigenvectors $|n\rangle$ with $\sum_{k=1}^{N} m_k n_k$ fixed give the same energy. To classify the degeneracy according to the representations of some symmetry algebra we make the following trick [5]. Let m be the least common multiple of all $m_k, k = 1, \ldots, N$, and let $l_k = \frac{m}{m_k}$. Moreover, let us put

$$n_k = q_k l_k + r_k, \qquad 0 \leqslant r_k \leqslant l_k - 1, \qquad k = 1, \ldots, N. \tag{3}$$

Then expression (2) for energy eigenvalues takes the form

$$E_n = H\left(m \sum_{k=1}^{N} q_k + \sum_{k=1}^{N} m_k (r_k + \sigma_k)\right). \tag{4}$$

Let $r \equiv (r_1, \ldots, r_N)$ be fixed; define X_r to be the subspace spanned by all vectors $|n\rangle$ such that $n_k = q_k l_k + r_k$; within X_r one can make an identification: $|n\rangle \equiv |q\rangle, q \equiv (q_1, \ldots, q_N)$. The whole space of states is then the orthogonal sum of $l_1 \cdot l_2 \cdots l_N$ subspaces X_r,

$$X = \bigoplus_r X_r. \tag{5}$$

When restricted to X_r, the energy spectrum of $\hat{H}$ takes a particularly simple form described by equation (4). The relevant symmetry algebra can be constructed as follows. First, we define the creation and annihilation operators $a_i^+, a_i, i = 1, \ldots, N$, as follows:

$$\begin{aligned} a_i^+|(n_1, \ldots, n_i, \ldots, n_N)\rangle &= \sqrt{n_i + 1}|(n_1, \ldots, n_i + 1, \ldots, n_N)\rangle \\ a_i|(n_1, \ldots, n_i, \ldots, n_N)\rangle &= \sqrt{n_i}|(n_1, \ldots, n_i - 1, \ldots, n_N)\rangle. \end{aligned} \tag{6}$$

On the other hand, in each X_r one can define another set of creation and annihilation operators:

$$\begin{aligned} b_{ri}^+|(q_1, \ldots, q_i, \ldots, q_N)\rangle &= \sqrt{q_i + 1}|(q_1, \ldots, q_i + 1, \ldots, q_N)\rangle \\ b_{ri}|(q_1, \ldots, q_i, \ldots, q_N)\rangle &= \sqrt{q_i}|(q_1, \ldots, q_i - 1, \ldots, q_N)\rangle. \end{aligned} \tag{7}$$

Obviously, the operators b_{ri}^+, b_{ri} are expressible in terms of basic ones a_i^+, a_i; the relevant formulae read

$$
\begin{aligned}
b_{ri} &= \left(\frac{N_i + l_i - r_i}{l_i}\right)^{\frac{1}{2}} \prod_{s=1}^{l_i} (N_i + s)^{-\frac{1}{2}} \cdot a_i^{l_i} \\
b_{ri}^+ &= \left(\frac{N_i - r_i}{l_i}\right)^{\frac{1}{2}} \prod_{s=1}^{l_i} (N_i - s + 1)^{-\frac{1}{2}} \cdot \left(a_i^+\right)^{l_i},
\end{aligned}
\tag{8}
$$

where $N_i \equiv a_i^+ a_i$ are the standard particle-number operators.

Now, the operators b_{ri}, b_{ri}^+ can be used in order to build symmetry algebra in each X_r. To this end let λ_α, $\alpha = 1, \ldots, N^2$, form the basis of $u(N)$ algebra. Then the operators

$$
\hat{\Lambda}_{r\alpha} \equiv \sum_{i,j=1}^{N} b_{ri}^+ (\lambda_\alpha)_{ij} b_{rj} \tag{9}
$$

have the following properties: (a) they are Hermitean, (b) commute with $\hat{H}$ in X_r, (c) obey $u(N)$ commutation rules in X_r.

Having constructed $u(N)$ symmetry algebra in each X_r one can define the symmetry operators in the whole space of states X. Let Q_r be the orthogonal projector on X_r. Then the operators

$$
\hat{\Lambda}_\alpha = \sum_r Q_r \hat{\Lambda}_{r\alpha} Q_r \equiv \sum_r \hat{\Lambda}_{r\alpha} Q_r \tag{10}
$$

span the $u(N)$ symmetry algebra in the total space of states X.

It remains to construct the projectors Q_r. It is easy to check that the following operators do the job:

$$
Q_r \equiv \prod_{i=1}^{N} P_i \qquad P_i = \frac{1}{l_i} \sum_{s=0}^{l_i - 1} \exp\left(\frac{2\mathrm{i}\pi s}{l_i}(N_i - r_i)\right). \tag{11}
$$

The creation and annihilation operators used above were constructed in a quite general abstract way. However, an interesting question is whether they can be expressed in terms of basic dynamical variables. In most cases, the answer to this question is positive although the resulting expressions are, in general, very complicated. Indeed, in almost all interesting cases (such as, for example, interacting particles, with or without spin, described by the natural Hamiltonian) the space of states carries an irreducible representation of the algebra of observables. The only possible exceptions we can see are the existence of superselection rules or the coexistence of discrete and continuum spectrum of the Hamiltonian. In the latter case, we must restrict ourselves to the discrete part of spectrum. The basic dynamical variables do have, in general, nonvanishing matrix elements between eigenstates corresponding to discrete and continuous spectra. This can be cured, for example, by inserting the projector on discrete subspace but it makes the whole procedure even more complicated.

As we have mentioned above, we are convinced that our assumptions (i)–(iii) are flexible enough to cover known cases of maximally superintegrable systems. Below we give a number of examples.

We start with the most obvious one—the harmonic oscillator with rational frequency ratios. The relevant Hamiltonian reads

$$
H = \sum_{k=1}^{N} \left(\frac{p_k^2}{2m_k} + \frac{m_k^2 l_k^2 \omega^2}{2} x_k^2\right) \equiv \sum_{k=1}^{N} H_k. \tag{12}
$$

Defining

$$\hat{I}_k = \frac{1}{\hbar\omega l_k} H_k, \tag{13}$$

we easily see that (i)–(iii) hold with all $\sigma_k = \frac{1}{2}$.

We can generalize this by adding the inverse square term [6]

$$H = \sum_{k=1}^{N} \left(\frac{p_k^2}{2m_k} + \frac{m_k^2 l_k^2 \omega^2}{2} x_k^2 + \frac{\alpha_k^2}{x_k^2} \right) \equiv \sum_{k=1}^{N} H_k; \tag{14}$$

then

$$\hat{I}_k = \frac{1}{2\hbar\omega l_k} H_k \tag{15}$$

and $\sigma_k = \frac{3}{4}$ for all k do the job.

Let us note in passing that it is sometimes claimed [6, 7] that, for some range of parameters, the spectrum of the Winternitz model is not equidistant; in contrast, if $\left|\frac{m_k\alpha_k^2}{\hbar^2}\right| < \frac{1}{8}$, the energy levels are organized in pairs which, in turn, are populated equidistantly. This is, however, not the case. The point is that it is not sufficient for the function to be square integrable and obey the relevant differential equation in order to become an eigenvector of the Hamiltonian. In contrast, the process of defining the selfadjoint operator from formal differential one is very subtle [8]. In the case of the Winternitz model, one easily checks that something goes wrong with the claimed energy spectrum by calculating the scalar product of eigenvectors corresponding to the energies of one pair of adjacent levels which is nonvanishing.

The next example is the two-dimensional Kepler problem:

$$H = \frac{\vec{p}^2}{2\mu} - \frac{\alpha}{r}. \tag{16}$$

The spectrum is discrete for negative energies and reads

$$E_n = \frac{-\mu\alpha^2}{2\hbar^2(n + |m| + 1)^2}, \tag{17}$$

m being the value of angular momentum. Therefore, we restrict ourselves to the subspace spanned by negative-energy eigenvectors of (16). Moreover, due to the appearance of $|m|$ in equation (17), we split this subspace further according to whether m is nonnegative or negative.

The dynamical symmetry behind is obtained by defining the Runge–Lenz vector [9]

$$A_i = \frac{1}{2\mu}\varepsilon_{ik}(p_k L + L p_k) - \frac{\alpha}{r} x_i. \tag{18}$$

One finds the following algebra:

$$[L, A_i] = \mathrm{i}\hbar\varepsilon_{ik} A_k \qquad [A_i, A_j] = -\frac{2\mathrm{i}\hbar}{\mu}\varepsilon_{ij} H L. \tag{19}$$

This is a quadratic algebra. However, in the subspace of constant negative energy it becomes $SO(3)$ algebra.

In order to see that (i)–(iii) are fulfilled, consider the subspace corresponding to $m \geqslant 0$ (the case $m < 0$ can be treated along the same lines). Define

$$\hat{I}_1 = \frac{1}{\hbar} L \qquad \hat{I}_2 = \frac{1}{\hbar}\left(\sqrt{\frac{-\mu\alpha^2}{2H}} - L \right). \tag{20}$$

These operators are well defined, selfadjoint and commuting; their spectra obey (i) with $\sigma_1 = 0, \sigma_2 = \frac{1}{2}$. Moreover,

$$H = \frac{-\mu\alpha^2}{2\hbar^2(\hat{I}_1 + \hat{I}_2)^2} \tag{21}$$

so that $m_1 = m_2 = 1$ in equation (1).

The three-dimensional Kepler problem can be dealt with in the same way. The orbital angular momentum $\vec{L}$ together with the Runge–Lenz vector [9]

$$\vec{A} = \frac{1}{2\mu}(\vec{p} \times \vec{L} - \vec{L} \times \vec{P}) - \frac{\alpha}{r}\vec{r} \tag{22}$$

forms the quadratic algebra:

$$\begin{aligned} [L_i, L_j] &= \mathrm{i}\hbar\varepsilon_{ijk}L_k \\ [L_i, A_j] &= \mathrm{i}\hbar\varepsilon_{ijk}A_k \\ [A_i, A_j] &= -\frac{2\mathrm{i}\hbar}{\mu}\varepsilon_{ijk}L_k H. \end{aligned} \tag{23}$$

Again, let us consider the negative energy subspace. Define

$$\begin{aligned} \hat{I}_1 &= \sqrt{\frac{\vec{L}^2}{\hbar^2} + \frac{1}{4}} + \frac{L_3}{\hbar} \\ \hat{I}_2 &= \sqrt{\frac{\vec{L}^2}{\hbar^2} + \frac{1}{4}} - \frac{L_3}{\hbar} \\ \hat{I}_3 &= \frac{1}{\hbar}\sqrt{-\frac{\mu\alpha^2}{2H}} - \frac{1}{2}(\hat{I}_1 + \hat{I}_2). \end{aligned} \tag{24}$$

Then (i)–(iii) are obeyed with $\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{2}$ and

$$H = -\frac{2\mu\alpha^2}{\hbar^2(\hat{I}_1 + \hat{I}_2 + 2\hat{I}_3)^2}. \tag{25}$$

Both harmonic oscillator and Kepler problem admit generalization consisting in replacing the Euclidean configuration space by the sphere [10] which preserves superintegrability. It is easy to check using the results of [10] that the relevant operators $\hat{I}_k$ can be constructed in a similar way as in the 'plane' case. We will not dwell on general case discussed in [10] (see, however, the remark below) but rather consider the simplest example of Kepler problem on two-dimensional sphere. Let λ be the curvature of this sphere. Following [10] we use the so-called gnomonic projection which is the projection onto the tangent plane from the centre of the sphere in the embedding space; the corresponding coordinates are denoted by x_1, x_2. Let p_1, p_2 be the canonical momenta. The Hamiltonian of our system reads

$$H = \frac{1}{2\mu}(\vec{\pi}^2 + \lambda L^2) - \frac{\alpha}{r}, \tag{26}$$

where

$$\vec{\pi} = \vec{p} + \tfrac{1}{2}\lambda(\vec{x}(\vec{x}\vec{p}) + (\vec{p}\vec{x})\vec{x}), \tag{27}$$

while L is the angular momentum:

$$L = \varepsilon_{ij}x_i p_j.$$

Again, as in the 'plane' case, the relevant Runge–Lenz vector can be constructed as

$$A_i = \frac{1}{2\mu}\varepsilon_{ij}(L\pi_j + \pi_j L) - \alpha\frac{x_i}{r}. \tag{28}$$

The second equation in (19) is now replaced by

$$[A_i, A_j] = \frac{2\mathrm{i}\hbar}{\mu}\varepsilon_{ij}L\left(-\frac{2H}{\mu} + \frac{\lambda}{\mu^2}\left(2L^2 + \frac{\hbar^2}{4}\right)\right). \tag{29}$$

The results of [10] allow us to find almost immediately the relevant $\hat{I}$'s:

$$\begin{aligned}\hat{I}_1 &= \frac{1}{\hbar}L\\ \hat{I}_2 &= \frac{1}{\hbar}\left(\sqrt{\frac{\hbar^2}{4} + \frac{1}{\lambda}\left(\mu H - \frac{\lambda\hbar^2}{8} + \sqrt{\left(\mu H - \frac{\lambda\hbar^2}{8}\right)^2 + \lambda\left(\alpha^2\mu^2 + \frac{\mu\hbar^2 H}{2}\right)}\right)} - L\right).\end{aligned} \tag{30}$$

Let us note in passing that our method applies to both models in an arbitrary number of dimensions. This is because the dynamical symmetries here are $SU(n)$ or $SO(n+1)$ and one can use the Gelfand–Tseytlin method to construct the operators $\hat{I}_k$.

As another example let us take the rational Calogero model [11]. It is also known to be maximally superintegrable on the quantum level [12]. The relevant Hamiltonian reads

$$H = -\frac{\hbar^2}{2}\sum_{k=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j}\left(\frac{\nu(\nu-\hbar)}{x_{ij}^2}\right) + \frac{\omega^2}{2}\sum_{i=1}^{N}x_i^2, \tag{31}$$

where $x_{ij} \equiv x_i - x_j$. The eigenvectors $\Phi_{\vec{n}}$ of the Hamiltonian (31) are parametrized by N-vectors with component being natural numbers, $\vec{n} = (n_1, n_2, \ldots, n_N), n_l \geqslant 0, l = 1, \ldots, N$. The relevant energy eigenvalue,

$$H\Phi_{\vec{n}} = E_{\vec{n}}\Phi_{\vec{n}}, \tag{32}$$

reads

$$E_{\vec{n}} = \omega\left(\hbar\sum_{l=1}^{N} l n_l + \frac{N}{2}\hbar + \frac{N(N-1)\nu}{2}\right). \tag{33}$$

The simplest way to obtain this formula is to use the results due to Nekrasov [13]. Namely, it has been shown in [13] that there exists a quantum mapping transforming the Calogero system into the one living on N-dimensional torus T^N; the Hamiltonian (31) is then transformed into the total momentum on T^N projected onto the subspace of positive partial momenta.

Once equation (33) is established, it is easy to check that (i) and (ii) are satisfied. To this end, it is sufficient to define I's by

$$\hat{I}_k\Phi_{\vec{n}} = n_k\Phi_{\vec{n}}. \tag{34}$$

One can consider more exotic systems such as, for example, Krall–Scheffer systems [14, 15]. Referring for details to [14], we mention only that these are superintegrable systems which are related to the families of orthogonal polynomials in two variables [16]. They fit into our scheme if we define the operators $\hat{I}_{1,2}$ as follows:

$$\hat{I}_1 Q_n^{(i)}(x, y) = \mathrm{i}Q_n^{(i)}(x, y) \qquad \hat{I}_2 Q_n^{(i)}(x, y) = (n - \mathrm{i})Q_n^{(i)}(x, y), \tag{35}$$

where the notation is taken from [14]. Then

$$[\hat{I}_1, \hat{I}_2] = 0 \tag{36}$$

and (see [14] for the definition of L playing the role of Hamiltonian)

$$L = \alpha(\hat{I}_1 + \hat{I}_2)^2 + (\beta - \alpha)(\hat{I}_1 + \hat{I}_2). \tag{37}$$

We believe that the above examples substantiate our claim that the definition of quantum superintegrability adopted in the present paper is general enough to cover most of the known superintegrable systems.

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References

[1] de Boer J, Harnisze F and Tjin T 1996 *Phys. Rep.* **272** 139
[2] Bonatsos P, Daskaloyannis C and Kokkotas K 1994 *Phys. Rev.* A **50** 3700
Bonatsos P, Daskaloyannis C, Kolokotronis P and Lenis D 1994 Symmetry algebra of the planar anisotropic quantum harmonic oscillator with rational ratio of frequencies *Preprint* hep-th/9402099
Bonatsos P, Daskaloyannis C, Kolokotronis P and Lenis D 1994 The symmetry algebra of the N-dimensional anisotropic quantum harmonic oscillator with rational ratios of frequencies and the Nilsson model *Preprint* hep-th/9411218
[3] Gonera C and Majewski M 2001 *Acta Phys. Pol.* B **32** 1167
[4] Jauch J M and Hill E L 1940 *Phys. Rev.* **57** 641
Mukunda N 1967 *Phys. Rev.* **155** 1383
Fradkin D M 1967 *Prog. Theor. Phys.* **37** 798
Cisneros A and McIntosh H V 1970 *J. Math. Phys.* **11** 870
Moshinsky M and Quesne C 1971 *J. Math. Phys.* **12** 1772
Moshinsky M, Seligman T H and Wolf K B 1972 *J. Math. Phys.* **13** 901
Louck J D, Moshinsky M and Wolf K B 1973 *J. Math. Phys.* **14** 692, 696
Moshinsky M, Patera J and Winternitz P 1975 *J. Math. Phys.* **16** 82
[5] Louck J D and Moshinsky M 1973 *J. Math. Phys.* **14** 692
[6] Winternitz P, Smorodinsky Ya, Uhlir M and Fris J 1967 *Sov. J. Nucl. Phys.* **4** 444
[7] Gravel S 2003 Superintegrability, isochronicity and quantum harmonic behavior *Preprint* math-ph/0310004
[8] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics* II (New York: Academic)
[9] Shiff L 1968 *Quantum Mechanics* (New York: McGraw-Hill)
[10] Higgs P W 1979 *J. Phys. A: Math. Gen.* **12** 309
Leemon H J 1979 *J. Phys. A: Math. Gen.* **12** 489
[11] Calogero F 1971 *J. Math. Phys.* **12** 419
[12] Kuznetsov V 1996 *Phys. Lett.* A **218** 212
Gonera C 1998 *Phys. Lett.* A **237** 365
[13] Nekrasov N 1997 On a duality in Calogero-Moser-Sutherland Systems *Preprint* hep-th/9707111
Ruijsenaars S N M 1987 *Commun. Math. Phys.* **110** 191
Ruijsenaars S N M 1988 *Commun. Math. Phys.* **115** 127
[14] Harnad J, Vinet L, Yermolayeva O and Zhedanov A 2001 *J. Phys. A: Math. Gen.* **34** 10619
[15] Harnad J and Yermolayeva O 2002 *Proc. of the Workshop in Superintegrability in Classical and Quantum Systems (CRM, Universite de Montreal)*
[16] Krall H and Sheffer J M 1967 *Ann. Math. Pure Appl.* **76** 325